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Spin chain sigma models with fermions

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Abstract

The complete one-loop planar dilatation operator of $\mathcal{N}=4$ supersymmetric Yang-Mills is isomorphic to the hamiltonian of an integrable $PSU(2,2|4)$ quantum spin chain. We construct the non-linear sigma models describing the continuum limit of the $SU(1|3)$ and $SU(2|3)$ sectors of the complete $\mathcal{N}=4$ chain. We explicitely identify the spin chain sigma model with the one for a superstring moving in $AdS_5 \times S^5$ with large angular momentum along the five-sphere.

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1 Introduction

The AdS/CFT correspondence is a fascinating proposal relating the weak coupling regime of a gauge theory to a strong coupling regime in string theory, and vice versa. A precise formulation of the correspondence would require complete access to the strong coupling regime of each theory and, in particular, a detailed understanding of the quantization of the string action in an $AdS_5 \times S^5$ background, which is indeed an involved problem. There is however a maximally supersymmetric plane-wave limit of the $AdS_5 \times S^5$ background for the IIB string [1], that can be quantized in the light-cone gauge [2]. On the gauge theory side this limit corresponds to focusing on local operators with a large R-symmetry charge J, of the form $\mathrm{Tr}(Z^J \ldots)$, where Z is one of the $\mathcal{N} = 4$ complex scalars [3]. On the gravity side these operators are described in terms of small closed strings with their center of mass moving with large angular momentum J along a circle in S^5 [4].

More general string configurations with several large angular momenta along S^5 were afterwards proposed to correspond to operators composed of the three $\mathcal{N} = 4$ complex scalars [5]. The energy of these semiclassical strings turned out to admit an analytic expansion on the effective parameter λ/J^2, with λ the 't Hooft coupling of the gauge theory, and thus suggested a direct comparison with the anomalous dimensions of large Yang-Mills operators. A serious difficulty for calculating the anomalous dimensions of operators with a large number of constituent fields, once quantum corrections are introduced, is operator mixing. The way out of this obstacle came from the deep observation that in the planar limit the one-loop dilatation operator of $\mathcal{N} = 4$ supersymmetric Yang-Mills [6, 7] is isomorphic to the hamiltonian of an integrable quantum spin chain [8, 9]. This identification allows to use the powerful Bethe ansatz technique in solving the spectrum of anomalous dimensions. Moreover, in the thermodynamic limit of very long spin chains the algebraic Bethe equations become integral equations, and therefore also the anomalous dimension of operators with very large spin turned reachable. Extending integrability to higher loops remains however an open problem, although strong arguments in favor exist for some sectors of $\mathcal{N} = 4$ [6], [10]-[13]. Using these techniques, the comparison of anomalous dimensions of gauge operators and energies of dual semiclassical strings has shown a perfect agreement up to two-loops [14]-[29]. It should however be stressed a disagreement starting at three-loops [15, 11, 13], which is currently one of the most intriguing questions on the AdS/CFT correspondence.

A promising path towards a more complete understanding of the correspondence between operators with large quantum numbers and semiclassical string solutions was opened in [30], where the action describing the continuum limit of the spin chain in the coherent state basis was directly compared with the dual string non-linear sigma model. As a first step, the continuum limit of the $SU(2)$ Heisenberg spin chain, associated to the two-spin holomorphic scalar sector, was shown to reproduce the action of strings moving with large angular momentum along an S^3 section of S^5 [30]. Equivalence of the spin chain and string non-linear sigma models was then proved at two-loop order [31]. The identification of both non-linear sigma models at the level of the action also implies the equivalence of fluctuations around the solutions, and therefore the matching holds beyond rigid strings. This approach was subsequently applied to more general sectors. The non-linear sigma model of the integrable $SU(3)$ spin chain, describing the three-spin holomorphic scalar sector, was found to correspond to a string moving with large angular momentum along S^5 [32, 33], while the non-compact $SL(2)$ chain, corresponding to semiclassical strings spinning in both AdS_5 and S^5, was considered in [33, 34]. The continuum limit of the more involved non-holomorphic scalar sector was shown to reproduce the dual string action on a phase space formulation [35].

It is important to extend the above studies to the continuum limit of gauge theory sectors associated to operators containing also fermions. In this way we should reproduce the dual superstring action and not only its bosonic part, as it was the case in the previous examples. A particularly interesting sector of the complete $\mathcal{N} = 4$ chain is that with spins in the fundamental representation of $SU(2|3)$ [10]. It describes operators composed of the three complex scalars and two of the sixteen gaugino components. This sector is closed at all orders in perturbation theory and already presents some of the main characteristics of the higher loop dynamics of the complete $\mathcal{N} = 4$ chain, which are absent in the simpler $SU(2)$ sector. Namely, the dilatation operator is part of the symmetry algebra and the number of sites of the chain is allowed to fluctuate [10]. In this paper we will derive the continuum limit of the $SU(2|3)$ chain. We will however restrict our analysis to contributions at leading order in the 't Hooft coupling. This will be enough to exhibit an interesting phenomenon present on the string theory side: the coupling of fermions to the RR five-form. The plan of the paper is the following. In section 2 we will introduce the problem. In section 3 we will construct the non-linear sigma model for the $SU(1|3)$ spin chain. In section 4 we will recover the spin chain sigma model from a large angular

momentum limit of the action for a superstring rotating in an S^3 section of S^5 (see also the related developments in [36]). Section 5 contains the continuum limit of the $SU(2|3)$ chain. We conclude with some directions of research in section 6.

2 $\mathcal{N} = 4$ operators with fermions

The calculation of the one-loop anomalous dimensions of $\mathcal{N} = 4$ supersymmetric Yang-Mills single trace operators, in the large N limit, can be very efficiently mapped to the problem of finding the spectrum of an integrable spin chain [8]. The vector space living at each site of the chain depends on the sector of Yang-Mills operators that we want to consider. For instance, operators composed of the three $\mathcal{N} = 4$ complex scalars map to a ferromagnetic Heisenberg $SU(3)$ spin chain, where at each site sits the fundamental representation. An important characteristic of this sector is that the anomalous dimensions of long operators only depend on the 't Hooft coupling through the combination $\frac{\lambda}{L^2}$, where L is the number of sites of the chain or, equivalently, the number of fields that compose the operator under study. The failure of this property would invalidate the derivation of the semiclassical coherent state action describing the continuum limit of the spin chain as performed in [30, 35].

We will be interested in the extension of the $SU(3)$ sector to include operators with fermions, but preserving the previous important property. In the scalar sector, this property does not hold generically for operators containing both complex scalar fields and their conjugates, or equivalently, fields with both positive and negative charges under the three Cartan generators J_i of the $SU(4)$ R-symmetry group [8, 35]. Guided by this fact, we will enlarge the holomorphic $SU(3)$ scalar sector by allowing for operators with insertions of those components of the gaugino field that, as is the case for the complex scalars, have positive J_i charges. This will reduce the sixteen components of the gaugino to two complex combinations λ_α, with $\alpha = 1, 2$. They form a Weyl spinor that transforms as $\left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right)$ under the Lorentz group, and is invariant under the $SU(3)$ subgroup of $SU(4)$ [10]. Thus this sector defines an $SU(2|3)$ spin chain based on the fundamental representation $\mathbf{3}|\mathbf{2}$, which accommodates the three scalar fields Z_i and the two fermions λ_α. The one-loop hamiltonian is [10]

$$H = \frac{\lambda}{8\pi^2} \sum_{l=1}^{L} \left(1 - SP_{l,l+1}\right) , \qquad (2.1)$$

where $SP_{l,l+1}$ denotes the super-permutation operator between two neighboring sites, l and $l+1$.

Our aim in this note will be to derive the non-linear sigma model associated to the $SU(2|3)$ spin chain, and to compare it to the action for strings on $AdS_5 \times S^5$, including the fermionic sector. This action is very involved, already at the classical level, due to the coupling of the fermions to the background RR five-form. However, it has been worked out explicitely for the fermionic quadratic terms in [37]. The quadratic approximation is valid when the fermionic excitations represent a small perturbation over a given bosonic background. Such configurations will map on the gauge theory side to operators with a large number of bosonic fields, and only a few fermionic insertions. Since the Lorentz group and the R-symmetry group commute, we expect that long $SU(2|3)$ operators with just a few fermionic insertions will be degenerated in the Lorentz index α. This implies that, as far as we will be interested in the comparison with the dual string action only at the fermionic quadratic level, it will be enough to restrict ourselves to the $SU(1|3)$ subsector of the $SU(2|3)$ chain. Notice that truncation to this subsector is consistent with the one-loop hamiltonian (2.1).

As a warming up step towards $SU(1|3)$, we will first consider the simpler case of $PSU(1|1) \subset SU(2|3)$. This subsector is associated to operators of the schematic form $\mathrm{Tr}(Z_1^J \lambda_1^{J'})$, with $J+J'=L$. At each site of this spin chain sits a two-dimensional vector space with one bosonic and one fermionic generator, $|b\rangle$ and $|f\rangle$, respectively. In order to construct a discrete sigma model for the spin chain we will introduce a set of coherent states at each site of the chain by applying an arbitrary $PSU(1|1)$ rotation to $|b\rangle$,

$$
\begin{aligned}
|n\rangle &= e^{i\xi S_y}\, e^{i\chi S_x}\, |b\rangle = \big(1 - \frac{1}{2}\,\zeta^*\zeta\big)|b\rangle + \zeta\,|f\rangle \ , \\
\langle n| &= \langle b|\, e^{iS_x\chi}\, e^{iS_y\xi} = \langle b|\big(1 - \frac{1}{2}\,\zeta^*\zeta\big) + \langle f|\,\zeta^* \ ,
\end{aligned} \tag{2.2}
$$

where S_x and S_y are the two odd generators of $PSU(1|1)$, ξ and χ are two Grassmann variables, and we have introduced the complex combination $\zeta = -\xi + i\chi$. The coherent states (2.2) verify $\langle n|n\rangle = 1$. They form an overcomplete basis, with the resolution of the identity

$$
\int d\zeta d\zeta^*\, |n\rangle\langle n| = \mathbb{1} \ . \tag{2.3}
$$

A path integral description of the partition function, as an integral over the overcomplete set of coherent states, shows the equivalence between the spin chain and the following

discrete sigma model (see for instance [38] for a complete derivation and details on the simpler case of the $SU(2)$ spin s chain)

$$S = -\int dt \left[i\langle \mathbf{n}|\frac{d}{dt}|\mathbf{n}\rangle + \langle \mathbf{n}|H|\mathbf{n}\rangle \right] , \tag{2.4}$$

where we have introduced the product of coherent states along the chain $|\mathbf{n}\rangle = |n_1 \cdots n_L\rangle$.

In order to derive the effective action describing the continuum limit of the spin chain, it is generically necessary to take into account quantum effects of short wavelength configurations [31]. However, when we are only interested in capturing the physics at first order in $\frac{\lambda}{L^2}$, it is enough to substitute in (2.4) finite differences between variables at neighboring sites by derivatives [38]. After an straightforward calculation we obtain

$$S = -\frac{L}{2\pi}\int d\sigma dt \left[i\zeta^*\partial_t\zeta + \frac{\lambda}{2L^2}\partial_\sigma\zeta^*\partial_\sigma\zeta \right] , \tag{2.5}$$

which is the action describing a non-relativistic fermion, with mass $m = \frac{L^2}{\lambda}$. This action reproduces, at first order in λ/L^2, the spectrum of fermionic fluctuations in the *pp*-wave limit [3].

3 The $SU(1|3)$ spin chain

The $SU(1|3)$ sector of $\mathcal{N} = 4$ Yang-Mills consists of operators composed of the three complex scalar fields and one component of the Weyl fermion. The derivation of the continuum limit of the associated $SU(1|3)$ spin chain follows the same steps as above. We will introduce a set of spin coherent states at each site of the chain by applying an arbitrary rotation to a bosonic eigenstate of the Cartan generators. Since such state is left invariant by an $SU(1|2)$ subgroup and just multiplied by a phase via an additional $U(1)$, the set of coherent states will be isomorphic to the quotient $SU(1|3)/SU(1|2) \times U(1)$. Hence to describe them we need four real variables and two Grassmann ones. An easy way to construct the $SU(1|3)$ coherent states is to consider first those for a $PSU(1|1)$ subsector and then apply to them an arbitrary $SU(3)$ rotation. The resulting states will have the same form (2.2) as before, but with $|b\rangle$ denoting now a $SU(3)$ coherent state (we will use the same notation and conventions as in [32]),

$$|b\rangle = \cos\theta\, \cos\psi\, e^{i\varphi}\,|1\rangle + \cos\theta\, \sin\psi\, e^{-i\varphi}\,|2\rangle + \sin\theta\, e^{i\phi}\,|3\rangle \quad , \tag{3.1}$$

where $\theta, \psi \in [0, \pi/2]$ and $\phi+\varphi, \phi-\varphi \in [0, 2\pi]$. The coherent states satisfy again $\langle n|n\rangle = 1$ and expand the identity in an expression analogous to (2.3).

As before, the spin chain system is equivalent to a discrete sigma model with action (2.4). The continuum limit of the chain is obtained by evaluating (2.4) over configurations that vary smoothly along the chain. The calculation is lengthy but simple, and we obtain

$$S = S_B + S_F \; . \tag{3.2}$$

The bosonic part of the action is the $SU(3)$ spin chain sigma model derived in [32, 33]

$$S_B = \frac{L}{2\pi}\int d\sigma\, dt\, \big[\, C_0 - \frac{\lambda}{2L^2}\, e^2 \,\big] \; , \tag{3.3}$$

where the Wess-Zumino term, C_0, is the time component of the 1-form

$$C = -i\langle b|d|b\rangle = \sin^2\theta\; d\phi + \cos^2\theta\; \cos(2\psi)\; d\varphi \; , \tag{3.4}$$

and we have defined

$$e^2 = \theta'^2 + \cos^2\theta\; \left(\psi'^2 + \sin^2(2\psi)\,\varphi'^2\right) + \sin^2\theta\; \cos^2\theta\; \left(\phi' - \cos(2\psi)\,\varphi'\right)^2 \; , \tag{3.5}$$

which represents energy density of the $SU(3)$ sigma model, $H = \frac{\lambda}{4\pi L}\int d\sigma\, e^2$. The fermionic piece of the action is given by

$$S_F = -\frac{L}{2\pi}\int d\sigma dt\, \left[\, i\zeta^* D_t\zeta + \frac{\lambda}{2L^2}\Big(D_\sigma\zeta^* D_\sigma\zeta - e^2\zeta^*\zeta\Big)\,\right] \; , \tag{3.6}$$

with $D\zeta = d\zeta - iC\zeta$. One more condition should be added to (3.2) in order to represent Yang-Mills operators. The trace required to define gauge invariant operators implies that we should only considered translationally invariant field configurations, *i.e.* $P_\sigma = 2\pi\mathbb{Z}$.

Definitions of the coherent states that differ by a phase should be considered equivalent. Therefore in order to parameterize the set of coherent states, we have to make a choice of global phase. Under a change in the definition $|n\rangle \to e^{i\alpha}|n\rangle$, we have $\zeta \to e^{i\alpha}\zeta$ and $C \to C + d\alpha$. Consistently, this transformation leaves invariant S_F and only changes the bosonic term S_B by an irrelevant total derivative. However, it is important to notice that a phase choice can not be fixed globally over the set of $SU(3)$ coherent states. In particular, the one implicit in (3.1) becomes singular at $\theta = \frac{\pi}{2}$ and $\psi = 0, \frac{\pi}{2}$. The appearance of the covariant derivatives in (3.6) is a consequence of the non-triviality of the line bundle $\{e^{i\alpha}|b\rangle\}$, contrary to the simpler $PSU(1|1)$ case.

4 The superstring action

In this section we will describe how the fermionic quadratic terms of the dual superstring action reproduce, after some large angular momentum limit, the spin chain results obtained

in the previous section. The fermionic part of the type IIB Green-Schwarz superstring action in $AdS_5 \times S^5$ expanded to quadratic order near a particular bosonic string solution (with a flat induced metric) was described in [37]. Choosing the κ-symmetry so that both Majorana-Weyl spinors in ten dimensions are equal, the quadratic fermionic lagrangian is

$$L = -2i\,\bar{\vartheta}\Big(\rho^a D_a + \frac{i}{2}\epsilon^{ab}\rho_a\Gamma_*\rho_b\Big)\vartheta \quad , \qquad D_a = \partial_a + \frac{1}{4}\omega_a^{AB}\Gamma_{AB} \; , \tag{4.1}$$

where ρ_a and ω_a^{AB} (with $a = 0, 1$) are projections of the $AdS_5 \times S^5$ gamma matrices and spin connection,

$$\rho_a = \partial_a X^M E_M^A\,\Gamma_A \quad , \qquad \omega_a^{AB} = \partial_a X^M \omega_M^{AB} \; , \tag{4.2}$$

and X^M denotes the coordinates on AdS_5 (when $M = (0, 6, 7, 8, 9)$) and S^5 (for $M = (1, 2, 3, 4, 5)$), E_M^A is the ten-vein, and Γ_A are the flat space ten-dimensional gamma matrices. The second term in (4.1) is a mass term, and has its origin in the coupling to the RR 5-form [37]; we have defined $\Gamma_* = i\Gamma_{06789}$, with $\Gamma_*^2 = \mathbb{1}$.

The action for relativistic bosonic strings rotating in S^5 has been shown to reproduce the $SU(3)$ spin chain sigma model (3.3) [32, 33]. Hence, in order to reproduce the fermionic piece of the $SU(1|3)$ spin chain action, we should consider such string solutions as the bosonic background for the quadratic action (4.1). However, to simplify the calculation while keeping at a general level, we will restrict to strings rotating in an S^3 section of S^5. These bosonic solutions map to large operators of $\mathcal{N} = 4$ supersymmetric Yang-Mills composed of two of the three complex scalars, and correspond to setting $\theta = 0$ in (3.3). The metric on $\mathbb{R}_t \times S^3$ is

$$ds^2 = -dt^2 + d\psi^2 + \cos^2\!\psi\, d\phi_1^2 + \sin^2\!\psi\, d\phi_2^2 \, . \tag{4.3}$$

We will make the gauge choice $t = \kappa\tau$, and associate the tangent space labels $A = 0, 1, 2, 3$ with the coordinates t, ψ, ϕ_1, ϕ_2 respectively. Then (with $\dot{X}^M = \partial_t X^M$ and $X'^M = \partial_\sigma X^M$ for the time and space derivatives)

$$\begin{aligned}
\rho_0 &= \kappa\Big[\Gamma_0 + \dot{\psi}\Gamma_1 + \dot{\phi}_1\cos\psi\Gamma_1 + \dot{\phi}_2\sin\psi\Gamma_3\Big] \, , \\
D_0 &= \kappa\Big[\partial_t + \frac{1}{2}\dot{\phi}_1\sin\psi\Gamma_{12} - \frac{1}{2}\dot{\phi}_2\cos\psi\Gamma_{13}\Big] \, , \\
\rho_1 &= \psi'\Gamma_1 + \phi_1'\cos\psi\Gamma_2 + \phi_2'\sin\psi\Gamma_3 \, , \\
D_1 &= \partial_\sigma + \frac{1}{2}\phi_1'\sin\psi\Gamma_{12} - \frac{1}{2}\phi_2'\cos\psi\Gamma_{13} \, .
\end{aligned} \tag{4.4}$$

While the spin chain action (3.6) is a one-loop result, the string action (4.1) captures arbitrary orders of the effective coupling constant. For comparing both actions, it is thus enough to take into account the effect of the rotating string background on the fermionic fluctuations at order λ/L^2. The energy of the bosonic string solutions that we are considering verifies [5]

$$E = \sqrt{\lambda}\,\kappa = L\left[1 + \mathcal{O}\Big(\frac{\lambda}{L^2}\Big)\right] , \tag{4.5}$$

where L denotes the total angular momentum. Since κ and $L/\sqrt{\lambda}$ only differ in subleading terms, and we are only interested in the leading dependence on the coupling constant, we can use κ instead of $L/\sqrt{\lambda}$ as the expansion parameter of the string action. A systematic expansion can be obtained after we distinguish between fast and slow movement of the rotating string solution [30, 35]. This is facilitated by the change of coordinates

$$\phi_1 = \alpha + \varphi \ , \quad \phi_2 = \alpha - \varphi \ . \tag{4.6}$$

The conjugated momentum of the variable α is the total angular momentum L. The limit of large L corresponds to strings rotating close to the speed of light in the α direction. This fast movement can be absorbed by shifting $\alpha \to \alpha + t$. After this redefinition, the string propagation verifies $\dot{X}^M = \mathcal{O}\big(\frac{1}{\kappa^2}\big)$, for $M \neq 0$ [30].

The spin chain action for the fermionic fluctuations (3.6) describes two-dimensional fermions with canonically normalized kinetic terms, coupled to a gauge field which carries the information about the bosonic background. Contrary, in the fermionic string action (4.1) the background is felt both on the matrices ρ_a and on the covariant derivatives D_a. For the comparison of both actions, the first step is to trivialize the matrices ρ_a by applying a series of rotations and concentrate the effect of the background only on D_a [1]. Let us start by writing the matrices ρ_a in terms of the variables (4.6), after the shift $\alpha \to \alpha + t$,

$$\begin{aligned}
\rho_0 &= \kappa\Big[\Gamma_0 + \dot{\psi}\Gamma_1 + (1+\dot{\alpha})(\cos\psi\Gamma_2 + \sin\psi\Gamma_3) + \dot{\varphi}(\cos\psi\Gamma_2 - \sin\psi\Gamma_3)\Big] \ , \\
\rho_1 &= \psi'\Gamma_1 + \alpha'(\cos\psi\Gamma_2 + \sin\psi\Gamma_3) + \varphi'(\cos\psi\Gamma_2 - \sin\psi\Gamma_3) \ .
\end{aligned} \tag{4.7}$$

We can now perform the necessary rotations $\rho_a \to S\rho_a S^{-1}$ as an expansion in $1/\kappa^2$, from which we will only retain the leading term. Applying two consecutive rotations $S_1 = e^{\frac{1}{2}p_1\Gamma_{23}}$ and $S_2 = e^{\frac{1}{2}p_2\Gamma_{21}}$, with

$$p_1 = \psi - \sin(2\psi)\dot{\varphi} \ , \quad p_2 = \dot{\psi} \ , \tag{4.8}$$

[1]The same process was followed in [39] to derive the spectrum of fermionic fluctuations around a particular circular string solution.

we obtain

$$\rho_0 = \kappa\Big[\Gamma_0 + \Big(1 - \frac{e^2}{2\kappa^2}\Big)\Gamma_2\Big] \quad , \qquad \rho_1 = \psi'\Gamma_1 - \sin(2\psi)\varphi'\,\Gamma_3 \; , \tag{4.9}$$

where we have used the Virasoro constraints, and e is the quantity defined in (3.5). Two further rotations lead to $\rho_0 = e\Gamma_0$ and $\rho_1 = e\Gamma_3$,

$$\begin{aligned} S_3 &= e^{\frac{1}{2}p_3\Gamma_{02}} \quad , \qquad \cosh p_3 = \frac{\kappa}{e} \quad , \qquad \sinh p_3 = -\frac{\kappa}{e} + \frac{e}{2\kappa} \, , \\ S_4 &= e^{\frac{1}{2}p_4\Gamma_{31}} \quad , \qquad \cos p_4 = -\frac{\sin(2\psi)\varphi'}{e} \quad , \qquad \sin p_4 = \frac{\psi'}{e} \, . \end{aligned} \tag{4.10}$$

The transformation of the covariant derivatives D_a under the previous rotations is tedious but straightforward to derive. The final result for the quadratic fermionic lagrangian (4.1) is

$$L = -2i\kappa^2\,\bar\Psi\,\Big[\Gamma_0\big(\partial_t + i\Pi A_t\big) - \frac{1}{\kappa}\Gamma_3\big(\partial_\sigma + i\Pi A_\sigma\big) + \Big(1 - \frac{e^2}{2\kappa^2}\Big)\Gamma_0\Gamma_3\Gamma_2\bar\Pi\Big]\,\Psi \; , \tag{4.11}$$

with $\Pi = -i\Gamma_0\Gamma_1\Gamma_2\Gamma_3$ and $\bar\Pi = \Gamma_6\Gamma_7\Gamma_8\Gamma_9$. The new spinor field is given by $\Psi = \sqrt{\frac{e}{\kappa}}\,S_4S_3S_2S_1\,\vartheta$. The factor $\kappa^{-\frac{1}{2}}$ in the normalization of the spinor has been introduced to compensate the effect of the S_3 rotation, such that Ψ remains finite in the large κ limit. We have also introduced

$$A_t = C_t + \frac{1}{2}(\dot p_4 - 1) \quad , \qquad A_\sigma = C_\sigma + \frac{1}{2}\,p_4' \; , \tag{4.12}$$

with C as defined in (3.4), with $\theta = 0$. This background field can be interpreted as a connection for the transformation $\Psi \to e^{\frac{i}{2}\Pi f}\Psi$, which is a symmetry of the action (4.11); choosing $f = t - p_4$ we obtain

$$A = C \; . \tag{4.13}$$

It was important in deriving (4.11) to separate the movement of the background S^3 string solution into fast and slow components, since this allowed us to perform a κ-expansion of the action. For consistency, in the time evolution of the fermions we should also distinguish between fast and slow oscillations. This can be achieved in the following way. As in [39], we split the ten-dimensional spinor Ψ into eigenstates of the projector $\frac{1}{2}(1+\bar\Pi)$, which commutes with all other operators. Given that the gamma matrices corresponding to two of the S^5 directions do not appear in (4.11), we can divide the eight real components of Ψ with the same $\bar\Pi$-eigenvalue into two sets of four which do not mix with each other. We can now interpret (4.11) as the lagrangian for a spinor with four

real components, and consider Γ_M as four-dimensional gamma matrices, with $M = 0, .., 3$. We will choose the representation $\Gamma_0 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\Gamma_j = i\begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix}$, with $\sigma_1 = \sigma_z$, $\sigma_2 = \sigma_x$, $\sigma_3 = \sigma_y$ the Pauli matrices. The dynamics defined by (4.11) is then consistent with taking Ψ to have the structure of a four-dimensional Majorana spinor, $\Psi = \begin{pmatrix} \xi \\ \sigma_y \xi^* \end{pmatrix}$. The last term in (4.11) gives mass to $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$, with $m = \pm\big(1 - \frac{e^2}{2\kappa^2}\big)$ for each component. The fast oscillations of the fermionic fluctuations are due to the contribution ± 1 to m, and can be subtracted from the positive frequency components by redefining $\xi \to e^{it}\xi$. After this shift, the equations of motion become

$$
\begin{aligned}
D_t\,\xi_1 + \frac{i}{\kappa} D_\sigma\,\xi_2 + \frac{ie^2}{2\kappa^2}\,\xi_1 &= 0\,, \\
D_t\,\xi_2 - \frac{i}{\kappa} D_\sigma\,\xi_1 - \frac{ie^2}{2\kappa^2}\,\xi_2 &= -2i\,\xi_2\,,
\end{aligned}
\tag{4.14}
$$

with $D = d - iC$. At leading order in $1/\kappa$ they imply

$$
\xi_2 = \frac{1}{2\kappa} D_\sigma\,\xi_1\ . \tag{4.15}
$$

Substituting in (4.11), the action becomes

$$
S = -\frac{\sqrt{\lambda}\kappa}{2\pi}\int d\sigma dt\,\left[i\xi_1^* D_t\xi_1 + \frac{1}{2\kappa^2}\Big(D_\sigma\xi_1^* D_\sigma\xi_1 - e^2\xi_1^*\xi_1 \Big)\right]\ . \tag{4.16}
$$

This action precisely coincides with the spin chain action (3.6) in the $\theta = 0$ sector, after we use the leading order relation $\kappa = L/\sqrt{\lambda}$. Notice that the term proportional to e^2 has its origin, from the string theory point of view, in the coupling of the fermions to the background RR 5-form.

From the ten-dimensional spinor Ψ we actually obtain four fermions with the same action (4.16). These fluctuations around the $\theta = 0$ sector, or equivalently, around a bosonic string rotating on a S^3 section of S^5, map to long Yang-Mills operators composed of two complex scalars with few fermionic insertions. From the sixteen gaugino components there are four of them positively charged under the two Cartan generators of the $SU(4)$ R-symmetry group that act non-trivially on two complex scalars. Following the arguments of section 2, we expect then a four-fold degeneracy also in the gauge theory side.

The comparison between the superstring action, including the fermionic part, and a dynamical system describing long Yang-Mills operators was also addressed in [36] following a different approach.

5 The $SU(2|3)$ spin chain

In this section we will derive the continuum limit of the complete $SU(2|3)$ spin chain. As we have explained in section 2, the $SU(2|3)$ sector of $\mathcal{N}=4$ is associated to operators composed of the three complex scalars and two complex fermions. We will choose the following parameterization for the coherent states in this sector,

$$|n\rangle = \big(1 - \frac{1}{2}\zeta_1^*\zeta_1 - \frac{1}{2}\zeta_2^*\zeta_2 + \frac{3}{4}\zeta_1^*\zeta_1\zeta_2^*\zeta_2\big)\big(|b\rangle + \zeta_1|f_1\rangle + \zeta_2|f_2\rangle\big)\ , \tag{5.1}$$

with $|b\rangle$ again the $SU(3)$ coherent state (3.1), $|f_1\rangle$ and $|f_2\rangle$ fermionic states belonging to the fundamental representation of $SU(2|3)$, and ζ_1 and ζ_2 two complex Grassmann variables. The term factored out in (5.1) insures $\langle n|n\rangle = 1$.

The action describing the continuum limit of the chain is obtained following the steps by now familiar. The bosonic part of this action gives again the $SU(3)$ non-linear sigma model (3.3). For the fermionic part we obtain

$$\begin{aligned} S_F \;=\; & -\frac{L}{2\pi}\int d\sigma dt \Big\{ i\zeta_1^* D_t\zeta_1 + i\zeta_2^* D_t\zeta_2 - iz^* D_t z + \\ & + \frac{\lambda}{2L^2}\Big[D_\sigma\zeta_1^* D_\sigma\zeta_1 + D_\sigma\zeta_2^* D_\sigma\zeta_2 - D_\sigma z^* D_\sigma z - e^2\big(\zeta_1^*\zeta_1 + \zeta_2^*\zeta_2 - 2z^* z\big)\Big]\Big\}\ , \end{aligned} \tag{5.2}$$

where we have defined

$$z = \zeta_1\zeta_2\ , \tag{5.3}$$

and the covariant derivatives are $D\zeta_j = d\zeta_j - iC\zeta_j$ (with $j=1,2$) and $Dz = dz - 2iCz$. The action (5.3) describes two non-relativistic fermions with quartic interactions. At the quadratic level this action coincides with (3.6) for each of the fields ζ_j. This confirms that it is enough to study the $SU(1|3)$ subsector when we are only interested in the quadratic fermionic terms. It would be very interesting to compare the quartic terms obtained here with the dual superstring action.

6 Conclusions

In this paper we have derived the non-linear sigma model arising from the continuum limit of the spin chains associated to the $SU(1|3)$ and $SU(2|3)$ sectors of $\mathcal{N}=4$ Yang-Mills. We have then explicitely recovered the $SU(1|3)$ spin chain sigma model from the non-linear sigma model for a superstring rotating in an S^3 section of S^5, showing thus that the identification of field and string theory actions also holds when fermions are included.

We have restricted our analysis to leading order in the 't Hooft coupling. In [10] the dilatation operator in the $SU(2|3)$ sector has been determined, and shown to be integrable, up to three-loops. It would be interesting then to extend our derivation to higher loops following [31], where the comparison between the spin chain and the string actions was performed to two-loops in the $SU(2)$ sector. At higher order in the 't Hooft coupling, the $SU(2|3)$ sector presents a feature characteristic of the complete $\mathcal{N}=4$ theory: the number of sites is allowed to fluctuate, due to the mixing of the three scalar fields into two fermions. This mixing has been argued to be absent in the thermodynamic limit of the chain [26], making the extension of our analysis to two-loops feasible. To address the important phenomenon of length fluctuation, it would be necessary to consider $1/L$ corrections. It should be noted however that the comparison of the $1/L$ corrections between the gauge and gravity sides is an open problem, even at leading order [40, 41]. A relevant result for understanding what is the reflect on the string side of the fluctuations in length of the spin chain could come from [42], where a certain conserved charge of generic fast moving strings has been conjectured to be associated with the spin chain length.

Another interesting direction to follow is analyzing the continuum limit of spin chains with less symmetries. Sigma models describing strings on orbifolds have recently been shown to match with the continuum limit of spin chains with twisted boundary conditions [43]. Further cases to study are the spin chain for $\mathcal{N}=2$ Yang-Mills [44], or chains connected to deformations of $\mathcal{N}=4$ Yang-Mills [45, 46]. This approach could provide a constructive way of obtaining the string duals of more generic gauge theories and, at the same time, understanding the behavior of string theory in more general backgrounds.

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